Optimal Voting Rules

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Collective Decision Problem

- n individuals must choose one among several alternatives
- Each individual has a private, cardinal, single-peaked preference over alternatives
- Monetary transfers are limited
- Designer's problem: find a mechanism that maximizes the sum of the individuals' expected utilities
  - strategy proof or dominant strategy incentive compatibility (DIC)
  - without transfers: DIC mechanisms must be voting mechanisms (depending only peaks)

Median in the Babylonian Talmud

> *If one (of three appraisers) says 100, and two say 200, or one says 200 and two say 100, then the majority rules. If one says 100, one says 80, and one says 120, then the judgment is 100.*
>
> *— Babylonian Talmud, Baba Bathra 107a*

> *Both the one who says 80, and the one who says 100, agree that the value is at most 100; the other, who says 120, is just one, and one does not prevail against two. And both the one who says 100, and the one who says 120 agree that the value is at least 100; the other, who says 80, is just one, and one does not prevail against two.*
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Single-Peaked Preferences

- Sidestep the Gibbard-Satterthwaite Impossibility Theorem
- Black (1948):
  - mechanism that selects median peak is Pareto optimal, anonymous, and dominant strategy incentive compatible (DIC)
- Moulin (1980):
  - any Pareto optimal, anonymous, and DIC mechanism is a generalized median

Generalized Median

- Moulin (1980): AN, PO and DIC mechanism with n voters
  $\Leftrightarrow$ generalized median with $(n-1)$ "phantoms"
  - distribute $(n-1)$ phantoms prior to voting among certain alternatives, and pick median of $(n-1)$ phantom and n real votes
  - generalized median = phantom distribution $\{\ell_k\}$ with $\sum \ell_k = n-1$
- Classical median with $n = 2m+1$ voters
  - place m phantoms at the left extreme, another m at the right extreme
- "Left-dictator" mechanism with $2m+1$ voters
  - place $2m$ phantoms at the left extreme alternative
- Two-thirds supermajority to adopt "reform" with $3m$ voters
  - place $2m-1$ phantoms at "status quo", the remaining m at "reform"

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Our Contribution

- We show that any generalized median can be implemented by modifying the successive voting procedure commonly used in European parliaments
- We derive incentive-compatible welfare-maximizing mechanism

Related Papers (Partial List)

- Literature following up Moulin (1980):
  - Barbera, Gul & Stacchetti (1993), Barbera & Jackson (1994), Ching (1997), Chatterji & Sen (2011)
  - Ehlers, Peters & Storcken (2002), Nehring & Puppe (2007)
  - Shummer & Vohra (2002), Dokow et al. (2012)
  - Saporiti (2009): single-crossing preferences
- Successive voting procedure
  - parliament voting: Rasch (2000)
  - public good provision: Bowen (1943), Green and Laffont (1979)
  - monetary committees: Riboni and Ruge-Murcia (2010)
- Optimal voting rules
  - two alternatives: Schmitz & Troger (2012), Azrieli & Kim (2014)
  - three alternatives: Borgers & Postl (2009)

Model Setup: One Dimensional, Private Values

- n agents choose one out of K alternatives: $\mathcal{K} = \{1, ..., K\}$
- Agent i observes private signal x_i: $(x_1, ..., x_n) \sim \Phi$ on $[\underline{x}, \overline{x}]^n$
- Utility $u^k(x_i)$: not necessarily increasing in x_i
- No monetary transfers

Model Setup: Single-Crossing Preferences

- Single-crossing utilities w.r.t. the order of alternatives:
  - $\forall k, l$ with $k < l$, exist cutoff $x^{l,k}$ with $u^k\left(x^{l,k}\right) = u^l\left(x^{l,k}\right)$ such that

$$\begin{cases} u^k\left(x_i\right) > u^l\left(x_i\right) & \text{if } x_i < x^{k,l} \\ u^k\left(x_i\right) < u^l\left(x_i\right) & \text{if } x_i > x^{k,l} \end{cases}$$

- Each alternative is the top alternative for some type:
  - for any $k \in \mathcal{K}$, there exists $x_i \in [\underline{x}, \bar{x}]$ such that

$$u^k\left(x_i\right) > \max_{l \in \mathcal{K}, l \neq k} u^l\left(x_i\right)$$

- Let $x^k \equiv x^{k-1,k}$. Then these assumptions imply that
  1. cutoffs are well ordered: $\underline{x} \equiv x^1 < ... < x^K < x^{K+1} \equiv \bar{x}$
  2. agents' preferences are single-peaked
- Designer maximizes sum of expected utilities subject to IC

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Incentive Compatibility

- We focus on deterministic direct mechanisms
- Direct mechanism $g : [\underline{x}, \overline{x}]^n \to \mathcal{K} = \{1, ..., K\}$
- A mechanism is dominant strategy incentive compatible (DIC) or strategy proof if for any player i and for any x_i, x_i' and x_{-i}:

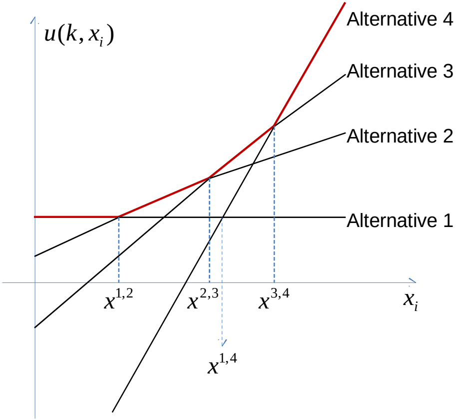
$$u^{g(x_i, x_{-i})}(x_i) \geq u^{g\left(x_i', x_{-i}\right)}(x_i)$$

Linear Example

- Linear utility: $u^k(x_i) = a_k + b_k x_i$ with $b_K > b_{K-1} > ... > b_1 \geq 0$
- Cutoff type $x^{l,k}$ is indifferent between l and k:

$$x^{l,k} \equiv \frac{a_l - a_k}{b_k - b_l}, \quad \text{and} \quad x^k \equiv x^{k-1,k} = \frac{a_{k-1} - a_k}{b_k - b_{k-1}}$$

Remark



- Agents' preferences are single-peaked
- But not all single-peaked preferences are compatible with linear preferences!

Example: First-Best Allocation Rule Is Not DIC

- Two alternatives $\{1, 2\}$ and two agents $\{i, -i\}$
- Planner is indifferent between 1 and 2 if

$$2a_1 + b_1 (x_i + x_{-i}) = 2a_2 + b_2 (x_i + x_{-i})$$

- First-best rule (maximizing sum of utilities) is monotone:

$$g(x_i, x_{-i}) = \begin{cases} 1 & \text{if} \quad (x_i + x_{-i})/2 \in [0, x^2) \\ 2 & \text{if} \quad (x_i + x_{-i})/2 \in [x^2, 1] \end{cases}$$

  with $x^2 \equiv (a_1 - a_2)/(b_2 - b_1)$
- Suppose $(x_i, x_{-i}) = \left(x^2 + \varepsilon, x^2 - 2\varepsilon\right)$: $g(x_i, x_{-i}) = 1$, but i can gain by reporting $x_i' > x^2 + 2\varepsilon$

Parliamentary Voting Procedures (Rasch, 2000)

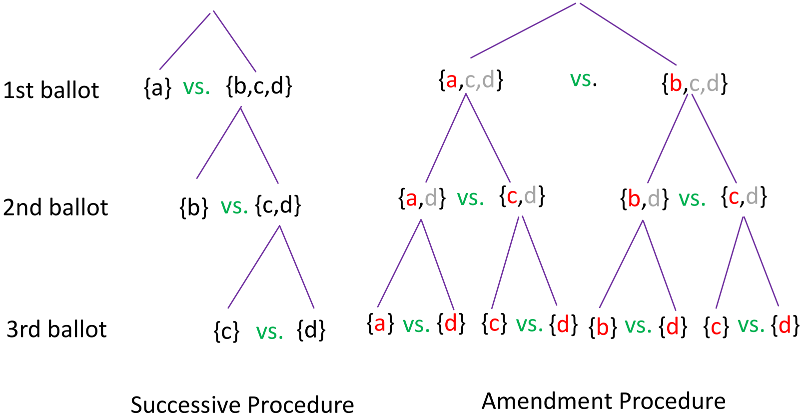
Successive Procedure (alternatives voted "one-by-one")	**Amendment Procedure** (alternatives voted "two-by-two")
Austria, Belgium, Czech Republic, Denmark, France, Germany, Greece, Hungary, Iceland, Ireland, Italy, Luxembourg, Netherlands, Norway, Poland, Portugal, Slovakia, Slovenia, Spain, European Parliament	Canada, Finland, Sweden, Switzerland, United Kingdom, USA

Successive Voting Procedure

- Voting on alternatives, one by one, in a pre-specified order
- If alternative 1 gets support of majority, it is adopted and voting ends. If alternative 1 fails, it is removed and the process proceeds to alternative 2
- If no alternative gains majority in earlier stages, the last two alternatives are paired and the one with majority support is adopted

Successive Voting Procedure

Consider 4 alternatives with voting order: a, b, c, d



Modified Successive Voting

- Alternatives are arranged in the natural order of $1, 2, ..., K$ under which preferences are single-peaked
  - natural if alternatives have numerical scale: public good, interest rate, tax, minimum wage

- The required majority is not constant across alternatives: threshold $\tau(k)$ for choosing alternative k (and stop) is decreasing
  - qualified majority
  - different voter thresholds for different local government charges
  - voter-approval requirements for local charges in California
    - non-tax charges (exempt from voter approval)
    - general tax (simple majority)
    - special tax (two-thirds)
    - property tax to finance infrastructure bonds (two-thirds), to finance school facility bonds (55%)

Sincere Voting Equilibrium

- A voting strategy for agent i is sincere if, at each stage, the agent votes in favor of the respective alternative if and only if it is the best (among the remaining alternatives) given his preferences

Proposition. *Sincere voting by all agents constitutes an ex-post perfect Nash equilibrium in a successive voting procedure with decreasing* $\tau(k)$. *It is the unique outcome that survives iterated elimination of (weakly) dominated strategies.*

Anonymity and Pareto Efficiency

- A mechanism g is anonymous if for any $x \in [0,1]^n$ $g(x_1, ..., x_n) = g\left(x_{\sigma(1)}, ..., x_{\sigma(n)}\right)$ where σ denotes any permutation of the set $\{1, ..., n\}$

- A mechanism g is Pareto efficient if for any $x \in [0,1]^n$ there is no alternative $k \in \mathcal{K}$ such that $u_i^k(x_i) \geq u_i^{g(x)}(x_i)$ for all i, with strict inequality for at least one agent

Dynamic Implementation of Static DIC Mechanisms

Theorem

1. *For any anonymous, Pareto efficient, and DIC mechanism g, there exists a decreasing threshold function $\tau^g(k)$ with $\tau^g(k) \leq n$ for any $k \in \mathcal{K}$ and $\tau^g(K) = 1$ such that, for any realization of types, the outcome of g coincides with the outcome in the sincere equilibrium of successive voting with thresholds $\tau^g(k)$.*
2. *Conversely, for any decreasing threshold $\tau(k)$ with $\tau(k) \leq n$ for any $k \in \mathcal{K}$ and $\tau(K) = 1$, there exists an anonymous, Pareto efficient, and DIC mechanism g^τ such that, for any realization of types, the outcome of g^τ coincides with the outcome in the sincere equilibrium of successive voting with thresholds $\tau(k)$.*

Coalition Interpretation of Generalized Median

- Simple median with $2m+1$ voters:
  - choose alternative k if
$$\#\left\{i|x_i < x^k\right\} \leq m \text{ and } \#\left\{i|x_i > x^{k+1}\right\} \leq m$$
  - choose alternative k such that
$$k = \min\left\{\tilde{k} : \#\{i|x_i \leq x^{\tilde{k}+1}\} \geq m+1\right\}$$
- Generalized median g with n voters and phantom distribution $\{\ell_k\}$
$$k = \min\left\{\tilde{k} : \#\{i|x_i \leq x^{\tilde{k}+1}\} \geq n - \sum_{m=1}^{\tilde{k}} \ell_m\right\}$$
- Successive voting with adopting threshold $\tau^g(k)$:
$$\tau^g(k) \equiv n - \sum_{m=1}^{k} \ell_m$$

Optimal Mechanisms

- Utilitarian planner
  - maximize the sum of the agents' expected utilities

$$\max_k \mathbb{E} \sum_i u^k(x_i)$$

  - number of feasible decreasing threshold functions:

$$\frac{(n+K-2)!}{(K-1)!(n-1)!}$$

- Combinatorial optimization problem

General Case

- Assumption A: Agents' signals are i.i.d. on $[0,1]$ according to F
- Assumption B: The function

$$\beta(k) = \frac{\left(u^{k}_{x>x^k} - u^{k-1}_{x>x^k}\right)}{\left(u^{k-1}_{x<x^k} - u^{k}_{x<x^k}\right) + \left(u^{k}_{x>x^k} - u^{k-1}_{x>x^k}\right)},\ k \geq 2$$

is decreasing, where

$$u^{l}_{x<x^k} = \mathbb{E}\left[u^{l}(x) \,|x < x^k\right] \text{ and } u^{l}_{x>x^k} = \mathbb{E}\left[u^{l}(x) \,|x > x^k\right]$$

- Intuition: rewrite definition of $\beta(k)$ as

$$\beta(k) \underbrace{\left(u^{k-1}_{x<x^k} - u^{k}_{x<x^k}\right)}_{\text{gain from switching to } k-1} + [1-\beta(k)] \underbrace{\left(u^{k-1}_{x>x^k} - u^{k}_{x>x^k}\right)}_{\text{loss from switching to } k-1} = 0$$

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Optimal Thresholds

Theorem

Under Assumptions A and B, the sincere equilibrium of the successive procedure with thresholds

$$\tau^*(k) = \begin{cases} \lceil n\beta(k+1) \rceil & \text{if } k < K \\ 1 & \text{if } k = K \end{cases}$$

implements the optimal anonymous, Pareto efficient, and DIC mechanism.

- Idea of proof, assuming $\tau^*(k-1) > \tau^*(k)$:
  - increasing $\tau^*(k)$ by 1 lower welfare $\Rightarrow \tau^*(k) \geq n\beta(k+1)$
  - decreasing $\tau^*(k-1)$ by 1 lower welfare $\Rightarrow \tau^*(k-1) \leq n\beta(k) + 1$
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Linear Case

- Assumption B': Distribution F has decreasing mean residual life ($\mathbb{E}[X - x|X > x] \searrow x$) and increasing reversed mean residual life ($\mathbb{E}[x - X|X < x] \nearrow x$)
  - implied by log-concave density; imply a decreasing $\beta(k)$

Corollary

Suppose utilities are linear and Assumptions A and B' hold. The optimal thresholds are

$$\tau^*(k) = \begin{cases} \lceil n\beta(k+1) \rceil & \text{if } k < K \\ 1 & \text{if } k = K \end{cases}$$

- Intuition: choose generalized median to make "mean voter" pivotal

$$\underbrace{n\beta(k+1)}_{\text{voters for } k}\mathbb{E}\left[X|X < x^{k+1}\right] + \underbrace{(n - n\beta(k+1))}_{\text{voters for } k+1}\mathbb{E}\left[X|X > x^{k+1}\right] = nx^{k+1}$$

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Correlated Types

- Linear utility setting:
  - underlying states $s \in \{1, ..., S\}$, with probability $\sum_{s=1}^{S} p_s = 1$
  - conditionally independent types: $x_s \sim F_s$, $s = 1, ..., S$
- We can still implement generalized median via successive voting
- Analysis of optimal voting rules becomes more complicated, because pivotal events reveal information about underlying state

Proposition. *Assume that $X_1 \leq_{lr} X_2 \leq_{lr} ... \leq_{lr} X_S$ and that each X_s satisfies Assumption B'. Suppose $\beta_S(k+1) \leq \beta_1(k)$ for all k. Then the optimal threshold $\tau^*(k)$ is the unique integer that satisfies the two necessary restrictions generated by local deviations.*

Large Societies

- Optimal mechanism attains the first-best welfare as $n \to \infty$
  - consider mechanism with fixed threshold $t : F\left(x^{k^*}\right) < t < F\left(x^{k^*+1}\right)$
- Linear utility
  - maximizing average utility = maximizing utility of the mean voter
  - $F(\mu)$-majority for adoption: $\beta\left(x^{k_\mu}\right) < F(\mu) < \beta\left(x^{k_\mu+1}\right)$
  - lognormal distribution with Gini $\eta \in [0.25, 0.55]$: $F(\mu) \in [0.58, 0.68]$
- Correlated types (example)
  - fixed threshold policy cannot attain the first-best
  - but a flexible threshold policy can

Public Good Provision

- n agents with preference for public good: $u_i = x_i G/\sqrt{n} + Z_i$
  - pubic good G, private consumption Z_i
  - factor $1/\sqrt{n}$ captures the negative effect of congestion
- Individual budget constraint: $M_i = Z_i + G^2/(2n)$
  - endowment M_i, shared quadratic cost of production $G^2/(2n)$
  - individual utility maximization: $G_i^* = \sqrt{n}x_i$
- Simple majority $G^{sm} = \sqrt{n}x_{sm}$, where x_{sm} is sample median
- Voting on successive increments, analogous to Bowen (1943):
  - require $1 - \beta(x)$ proportional support for further increase over $\sqrt{n}x$
  - with large population, $G^{s\mu} = \sqrt{n}x_{s\mu}$, where $x_{s\mu}$ is sample mean

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Concluding Remarks

- We characterize constrained efficient DIC mechanisms
  - single-crossing preferences but no transfers
  - our characterization enables a systematic choice among Pareto-efficient mechanisms
  - implement static Pareto efficient DIC mechanisms via a dynamic voting procedure
- Bayesian incentive compatibility? stochastic mechanisms?
- Two or more dimensions: work in progress